

A completely monotonic function used in an inequality of Alzer*

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Abstract

The function $G(x) = (1 - \ln x / \ln(1 + x)) x \ln x$ has been considered by Alzer, Qi and Guo. We prove that G' is completely monotonic by finding an integral representation of the holomorphic extension of G to the cut plane. A main difficulty is caused by the fact that G' is not a Stieltjes function.

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1 Introduction and results

In a recent paper [1], Alzer proved a number of inequalities involving the volume of the unit ball in \mathbb{R}^n ,

$$\Omega_n = \frac{\pi^{n/2}}{\Gamma(1 + n/2)}, \quad n = 1, 2, \dots \quad (1)$$

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That paper contains many references to earlier results about Ω_n . We mention in particular that Anderson and Qiu [2] proved that the sequence $f(n) = \Omega_n^{1/(n \log n)}$, $n \geq 2$ is strictly decreasing and converges to $e^{-1/2}$. It is therefore of interest to study the function

$$f(x) = \left(\frac{\pi^{x/2}}{\Gamma(1 + x/2)} \right)^{1/(x \ln x)}, \quad (2)$$

and in [9] the authors have given an integral representation of $\log f(x+1)$, $x > 0$ by considering its holomorphic extension to the cut plane $\mathcal{A} = \mathbb{C} \setminus (-\infty, 0]$. From this representation it has been possible to deduce that $f(n+2)$ is a Hausdorff moment sequence, in particular decreasing and convex.

The papers [2] and [3] have also been an inspiration for several papers about the functions

$$F_a(x) = \frac{\ln \Gamma(x+1)}{x \ln(ax)}, \quad x > 0, a > 0, \quad (3)$$

see [1],[7],[8],[9],[10],[11]. In particular, [9] contains an integral representation of the meromorphic extension of F_a to \mathcal{A} . From this representation it is possible to deduce that F_a is a Pick function if and only if $a \geq 1$. The relation between F_a and f is given by

$$\log f(z+1) = \frac{\ln \sqrt{\pi}}{\text{Log}(z+1)} - \frac{1}{2} F_2 \left(\frac{z+1}{2} \right).$$

Alzer found the best constants a^*, b^* such that for all $n \geq 2$

$$\exp \left(\frac{a^*}{n(\log n)^2} \right) \leq f(n)/f(n+1) < \exp \left(\frac{b^*}{n(\log n)^2} \right). \quad (4)$$

In the proof of this result Alzer considered the function

$$G(x) = \left(1 - \frac{\ln x}{\ln(1+x)} \right) x \ln x, \quad (5)$$

and in [1, Lemma 2.3] it was proved that $2/3 < G(x) < 1$ for $x \geq 3$. Qi and Guo observed in [11] that G is strictly increasing on $(0, \infty)$ with $G((0, \infty)) = (-\infty, 1)$ and that $G(3) > 2/3$, which gave a new proof of the inequality $2/3 < G(x) < 1$. Furthermore, in [11, Remark 4] it was conjectured that

$$(-1)^{k-1} G^{(k)}(x) > 0 \quad \text{for } x > 0, k = 1, 2, \dots, \quad (6)$$

or equivalently that G' is a completely monotonic function.

The main goal of this paper is to prove this conjecture. We do this by considering G as a holomorphic function in the cut plane \mathcal{A} . We put

$$G(z) = \left(1 - \frac{\text{Log } z}{\text{Log}(1+z)} \right) z \text{Log } z, \quad (7)$$

where $\text{Log } z = \ln |z| + i \text{Arg } z$ is the principal logarithm in \mathcal{A} and $-\pi < \text{Arg } z < \pi$ for $z \in \mathcal{A}$.

Using the same Cauchy integral formula technique as in the paper [9], we shall establish the following theorem.

Theorem 1.1 *The function G from (7) has the representation*

$$G(z) = 1 - \int_0^\infty \frac{\rho(t)}{z+t} dt, \quad z \in \mathcal{A}, \quad (8)$$

where

$$\rho(t) = \begin{cases} -\frac{t \ln((1-t)/t^2)}{\ln(1-t)}, & \text{if } 0 < t < 1, \\ -\frac{t(\ln((t-1)/t))^2}{(\ln(t-1))^2 + \pi^2}, & \text{if } 1 < t < \infty. \end{cases} \quad (9)$$

Notice that $\rho(1^-) = \rho(1^+) = -1$ so that ρ is continuous on the positive half-line. It is decreasing from ∞ to -1 on the interval $(0, 1)$ with $\rho'(1^-) = -1$, and increasing from -1 to 0 on the interval $(1, \infty)$ with $\rho'(1^+) = \infty$. We have $\rho((\sqrt{5}-1)/2) = 0$. Notice also that ρ is integrable over $(0, \infty)$ because of the asymptotics

$$\rho(t) \sim -2 \ln t \text{ for } t \rightarrow 0^+; \quad \rho(t) \sim -\frac{1}{t(\ln t)^2} \text{ for } t \rightarrow \infty.$$

The graph of ρ is shown in Figure 1.

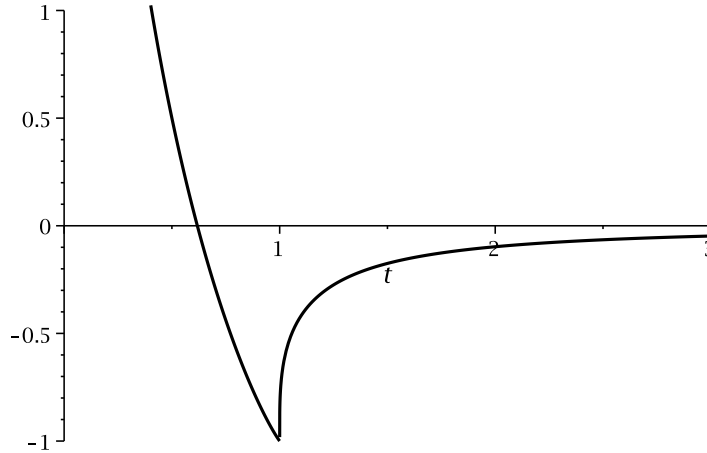


Figure 1: The graph of ρ

Since ρ assumes positive and negative values, G as well as $1 - G$ are not Stieltjes functions, but nevertheless $1 - G$ turns out to be completely monotonic, because it is the Laplace transform of a positive function, as described in the

following theorem. In particular, G' is completely monotonic so (6) holds. For properties about completely monotonic functions and Stieltjes functions we refer to [6] and [13].

Theorem 1.2 *For $\Re z > 0$ the function $1 - G$ has the representation*

$$1 - G(z) = \int_0^\infty e^{-zs} \varphi(s) ds, \quad (10)$$

where

$$\varphi(s) = \int_0^\infty e^{-st} \rho(t) dt > 0 \text{ for } s \geq 0. \quad (11)$$

The graph of φ is given in Figure 2.

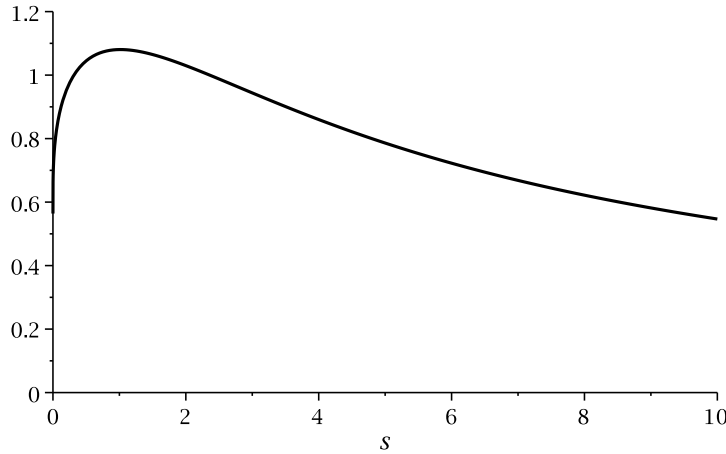


Figure 2: The graph of φ

The function φ given in (11) is continuous and bounded on $[0, \infty)$, but it is not integrable because $1 - G(x) \rightarrow \infty$ for $x \rightarrow 0^+$.

Setting $z = a + it$ in (10) with $a > 0$ we get:

Corollary 1.3 (i) *For each $a > 0$*

$$1 - G(a + it) = \int_0^\infty e^{-its} e^{-as} \varphi(s) ds, \quad t \in \mathbb{R} \quad (12)$$

is an analytic positive definite function of t , and it is the Fourier transform of

$$e^{-as} \varphi(s) 1_{[0, \infty)}(s). \quad (13)$$

(ii) $G(a + it) - G(a)$ is a continuous negative definite function of t for each $a > 0$. In particular

$$\Re G(a + it) \geq G(a), \quad a > 0, t \in \mathbb{R}. \quad (14)$$

(iii) $G(a + it)$ is a continuous negative definite function of t for $a \geq 1$.

Concerning continuous positive and negative definite functions we refer to e.g. [6].

Letting $a \rightarrow 0^+$ in (12), we formally get that $1 - G(it)$ is the Fourier transform of $\varphi(s)1_{[0,\infty)}(s)$. This is true in the L^2 -sense because of Plancherel's theorem. In fact, we have

Proposition 1.4 *The function φ in (11) is square integrable and*

$$\lim_{a \rightarrow 0^+} \int_{-\infty}^{\infty} |1 - G(a + it)|^2 \frac{dt}{2\pi} = \int_{-\infty}^{\infty} |1 - G(it)|^2 \frac{dt}{2\pi} = \int_0^{\infty} \varphi^2(s) ds. \quad (15)$$

The function G is one-to-one when considered on the positive real line. It is shown below that G is conformal when defined in a sector containing the positive real line. We put

$$S(a, b) = \{z \neq 0 \mid a < \text{Arg } z < b\}.$$

Proposition 1.5 *The function $G : S(-\pi/3, \pi/3) \rightarrow \mathbb{C}$ is a conformal mapping.*

Based on computer experiments it seems that G is conformal in the right half plane, but we have not been able to verify this. On the other hand, $G : \mathcal{A} \rightarrow \mathbb{C}$ is not conformal.

2 Proof of the properties of G

In the first lemma the behaviour of G close to zero and infinity is investigated.

Lemma 2.1 *We have*

(i) *There exist constants $A, B > 0$ such that*

$$|G(z)| \leq A |\text{Log } z| + B |\text{Log } z|^2 \text{ for } z \in \mathcal{A}, |z| \leq 1/2,$$

(ii) *$zG(z) \rightarrow 0$ for $z = \varepsilon e^{i\theta}, \varepsilon \rightarrow 0$, uniformly for $-\pi < \theta < \pi$.*

(iii) *There exists a constant $C > 0$ such that*

$$|1 - G(z)| \leq C/|z| \text{ for } z \in \mathcal{A}, |z| \geq e,$$

(iv) *$G(z) \rightarrow 1$ for $z = Re^{i\theta}, R \rightarrow \infty$, uniformly for $-\pi < \theta < \pi$.*

Proof. We have for $z \in \mathcal{A}$

$$G(z) = \frac{z}{\operatorname{Log}(1+z)} (\operatorname{Log}(1+z) \operatorname{Log} z - (\operatorname{Log} z)^2),$$

hence for $|z| \leq 1/2$

$$|G(z)| \leq \max_{|z| \leq 1/2} \left| \frac{z}{\operatorname{Log}(1+z)} \right| \left(|\operatorname{Log} z| \max_{|z| \leq 1/2} |\operatorname{Log}(1+z)| + |\operatorname{Log} z|^2 \right),$$

which shows (i).

(ii) follows from (i) since $z(\operatorname{Log} z)^n \rightarrow 0$ for $n \geq 1$ and $|z| = \varepsilon \rightarrow 0$.

To see (iii), we note that the power series (in $1/z$)

$$\operatorname{Log}(1 + 1/z) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{nz^n}, \quad |z| > 1$$

yields

$$|\operatorname{Log}(1 + 1/z)| \leq \sum_{n=1}^{\infty} 1/|z|^n = \frac{1}{|z| - 1} \leq \frac{1}{e - 1}, \quad |z| \geq e. \quad (16)$$

The power series also yields

$$z \operatorname{Log}(1 + 1/z) = 1 + \alpha(z)/z, \quad |\alpha(z)| \leq \frac{e}{2(e - 1)}, \quad |z| \geq e. \quad (17)$$

Note also that $|\operatorname{Log} z| \geq 1$ for $z \in \mathcal{A}, |z| \geq e$.

Writing

$$\frac{\operatorname{Log} z}{\operatorname{Log}(1+z)} = 1 + \beta(z) \operatorname{Log}(1 + 1/z),$$

with

$$\beta(z) = \frac{-1}{\operatorname{Log}(1+z)}$$

we find for $z \in \mathcal{A}, |z| \geq e$

$$|\beta(z)| = \frac{1}{|\operatorname{Log}(1+z)|} \leq \frac{1}{\ln|1+z|} \leq \frac{1}{\ln(|z| - 1)} \leq \frac{1}{\ln(e - 1)}. \quad (18)$$

Finally, since

$$G(z) = (z \operatorname{Log}(1 + 1/z)) \frac{\operatorname{Log} z}{\operatorname{Log}(1+z)} = (1 + \alpha(z)/z)(1 + \beta(z) \operatorname{Log}(1 + 1/z)),$$

we see that

$$z(G(z) - 1) = \alpha(z) + \beta(z)z \operatorname{Log}(1 + 1/z) + \alpha(z)\beta(z) \operatorname{Log}(1 + 1/z),$$

which by (17) and (18) is bounded by some constant $C > 0$ for $|z| \geq e$, showing (iii). Property (iv) follows immediately from (iii). \square

Proof of Theorem 1.1 For fixed $z \in \mathcal{A}$ we choose ε and R such that $0 < \varepsilon < |z| < R$ and consider the positively oriented contour $\gamma(\varepsilon, R)$ in \mathcal{A} consisting of the half-circle $z = \varepsilon e^{i\theta}$, $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and the half-lines $z = x \pm i\varepsilon$, $x \leq 0$ until they cut the circle $|z| = R$, which closes the contour at the points $-R(\varepsilon) \pm i\varepsilon$, where $0 < R(\varepsilon) \rightarrow R$ for $\varepsilon \rightarrow 0$. By Cauchy's integral theorem we have

$$G(z) = \frac{1}{2\pi i} \int_{\gamma(\varepsilon, R)} \frac{G(w)}{w - z} dw. \quad (19)$$

Letting ε tend to zero, the contribution corresponding to the half-circle with radius ε tends to 0 by (ii) of Lemma 2.1.

Concerning the boundary behaviour of G on the negative real line we obtain

$$G(t+i0) := \lim_{\varepsilon \rightarrow 0^+} G(t+i\varepsilon) = \begin{cases} \left(1 - \frac{\ln(-t)+i\pi}{\ln|1+t|+i\pi}\right) t(\ln(-t) + i\pi), & \text{if } t < -1 \\ \left(1 - \frac{\ln(-t)+i\pi}{\ln(1+t)}\right) t(\ln(-t) + i\pi), & \text{if } -1 < t < 0. \end{cases}$$

Note that $G(t+i0)$ is continuous at $t = -1$ with value $-i\pi$. Using that $G(\bar{z}) = \overline{G(z)}$, (19) yields

$$G(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{G(Re^{i\theta})}{Re^{i\theta} - z} Re^{i\theta} d\theta + \frac{1}{\pi} \int_{-R}^0 \frac{\Im G(t+i0)}{t - z} dt. \quad (20)$$

In the last integral we replace t by $-t$ and use that $(-1/\pi)\Im G(-t+i0) = -\rho(t)$. Letting $R \rightarrow \infty$ and using (iv) of Lemma 2.1, we finally get (8). \square

Remark 2.2 Feng Qi has kindly informed us about the following elementary proof of the observation that $1 - G$ is not a Stieltjes function. In fact, if it were, then also $h(x) = 1/(x(1 - G(x)))$ would be a Stieltjes function by the Stieltjes-Reuter-Itô Theorem, cf. [12], [4] or [5, p.25]. In particular, h will be decreasing, which is contradicted by the simple fact that $1 = h(1) < h(2) = 1.02 \dots$.

Proof of Theorem 1.2 The formulas (10) and (11) follow immediately from Theorem 1.1 and it remains to prove that φ is positive. Let $t_0 = (\sqrt{5} - 1)/2$. Then $\rho(t) > 0$ for $0 < t < t_0$ and $\rho(t) < 0$ for $t_0 < t < \infty$ and hence

$$A = \int_0^{t_0} \rho(t) dt > 0, \quad B = \int_{t_0}^{\infty} \rho(t) dt < 0. \quad (21)$$

Using this notation we get

$$\begin{aligned} \varphi(s) &= \int_0^{t_0} e^{-st} \rho(t) dt + \int_{t_0}^{\infty} e^{-st} \rho(t) dt \\ &\geq \int_0^{t_0} e^{-st_0} \rho(t) dt + \int_{t_0}^{\infty} e^{-st_0} \rho(t) dt = (A + B)e^{-st_0}. \end{aligned}$$

In the following lemma it is established that $A + B > 0$, and hence $\varphi(s) > 0$ for all $s \geq 0$. \square

Lemma 2.3

$$\int_0^\infty \rho(t) dt > 0.$$

Proof. We first establish

$$\int_0^1 \rho(t) dt > \frac{\pi^2}{6} - \frac{1}{2}. \quad (22)$$

Since

$$\sum_{n=0}^{\infty} t^n \int_0^1 \binom{x}{n} dx = \int_0^1 (1+t)^x dx = \left[\frac{(1+t)^x}{\ln(1+t)} \right]_0^1 = \frac{t}{\ln(1+t)}$$

we obtain the power series expansion

$$\frac{t}{\ln(1+t)} = 1 + \sum_{n=1}^{\infty} b_n t^n, \quad |t| < 1; \quad b_n = \int_0^1 \binom{x}{n} dx. \quad (23)$$

The numbers b_n are sometimes called the Cauchy numbers. Note that for $n \geq 1$

$$0 < (-1)^{n-1} b_n = \int_0^1 \frac{x(1-x) \cdots (n-1-x)}{n!} dx \leq \frac{1}{n} \int_0^1 x dx = \frac{1}{2n}. \quad (24)$$

By (23) we get

$$\begin{aligned} \int_0^1 \rho(t) dt &= -\frac{1}{2} - 2 \int_0^1 \ln t dt + 2 \sum_{n=1}^{\infty} (-1)^{n-1} b_n \int_0^1 t^n \ln t dt \\ &= \frac{3}{2} - 2 \sum_{n=1}^{\infty} (-1)^{n-1} \frac{b_n}{(n+1)^2} \end{aligned}$$

and hence using (24)

$$\begin{aligned} \int_0^1 \rho(t) dt &> \frac{3}{2} - \sum_{n=1}^{\infty} \frac{1}{n(n+1)^2} = \frac{3}{2} - \sum_{n=1}^{\infty} \frac{1}{n+1} \left(\frac{1}{n} - \frac{1}{n+1} \right) \\ &= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{(n+1)^2} = \frac{\pi^2}{6} - \frac{1}{2}. \end{aligned}$$

We next show that

$$\int_1^2 \rho(t) dt > -\frac{1}{12} - \frac{2(1 + \ln 2) \ln 2}{\pi^2}, \quad (25)$$

by using the rough estimate

$$\int_1^2 \rho(t) dt > -\frac{1}{\pi^2} \int_1^2 t(\ln(1 - 1/t))^2 dt$$

and

$$\int_1^2 t(\ln(1 - 1/t))^2 dt = \int_1^2 (t(\ln(t - 1))^2 + t(\ln t)^2 - 2t \ln(t - 1) \ln t) dt.$$

The integral of the first two terms can be calculated because

$$\int t(\ln t)^2 dt = \frac{t^2}{2} \left((\ln t)^2 - \ln t + \frac{1}{2} \right),$$

and for the integral of the third term we have

$$\begin{aligned} 2 \int t \ln(t - 1) \ln t dt &= \left(t^2 \ln t - \frac{1}{2} t^2 + \frac{1}{2} \right) \ln(t - 1) - \frac{1}{2} t^2 \ln t + \frac{1}{2} t^2 \\ &\quad - t \ln t + \frac{3}{2} t + \text{dilog}(t), \end{aligned}$$

where

$$\text{dilog}(t) = \int_1^t \frac{\ln x}{1 - x} dx.$$

Since

$$\text{dilog}(2) = - \int_0^1 \frac{\ln(1 + u)}{u} du = - \sum_{n=0}^{\infty} \int_0^1 (-1)^n \frac{u^n}{n + 1} du = -\frac{\pi^2}{12},$$

this leads to the expression in (25).

We finally show

$$\int_2^{\infty} \rho(t) dt > -\frac{1}{2}. \quad (26)$$

Squaring the power series for $\ln(1 - u)$ yields

$$(\ln(1 - u))^2 = u^2 \sum_{n=0}^{\infty} c_n u^n, \quad |u| < 1; \quad c_n = \sum_{k=0}^n \frac{1}{(k + 1)(n + 1 - k)}. \quad (27)$$

The relation $0 < c_n \leq 1$ for all n is proved in Lemma 2.5 below. Therefore, and using (27) with $u = 1/t$ it follows that

$$\begin{aligned}
\int_2^\infty \rho(t) dt &= - \int_2^\infty \sum_{n=0}^\infty \frac{c_n}{t^{n+1}((\ln(t-1))^2 + \pi^2)} dt \\
&> - \int_2^\infty \left(\sum_{n=0}^\infty \frac{1}{t^{n+1}} \right) \frac{dt}{(\ln(t-1))^2 + \pi^2} \\
&= - \int_2^\infty \frac{dt}{(t-1)((\ln(t-1))^2 + \pi^2)} \\
&= - \frac{1}{\pi} \int_0^\infty \frac{dx}{1+x^2} = - \frac{1}{2}.
\end{aligned}$$

Combining (22), (25) and (26) we get

$$\int_0^\infty \rho(t) dt > \frac{\pi^2}{6} - \frac{1}{2} - \frac{1}{12} - \frac{2(1+\ln 2)\ln 2}{\pi^2} - \frac{1}{2} \simeq 0.3238 > 0$$

and the lemma is proved. \square

Remark 2.4 *A numerical computation yields*

$$\varphi(0) = \int_0^\infty \rho(t) dt \simeq 0.5192.$$

Lemma 2.5 *The numbers*

$$c_n = \sum_{k=0}^n \frac{1}{(k+1)(n+1-k)}, \quad n \geq 0$$

can be written in the form

$$c_n = \frac{2\mathcal{H}_{n+1}}{n+2},$$

where $\mathcal{H}_n = \sum_{k=1}^n 1/k$ is the n 'th harmonic number. Moreover,

$$c_{n-1} - c_n = \frac{2(\mathcal{H}_n - 1)}{(n+1)(n+2)} \geq 0,$$

whence $1 = c_0 = c_1 > c_2 > c_3 \dots$

Proof. By definition we have

$$\begin{aligned}
c_n &= \sum_{k=0}^n \int_0^1 x^k dx \int_0^1 y^{n-k} dy = \int_0^1 \int_0^1 \frac{x^{n+1} - y^{n+1}}{x-y} dx dy \\
&= 2 \int_0^1 \left(\int_0^x \frac{x^{n+1} - y^{n+1}}{x-y} dy \right) dx \\
&= 2 \int_0^1 x^{n+1} dx \int_0^1 \frac{1-t^{n+1}}{1-t} dt = \frac{2\mathcal{H}_{n+1}}{n+2}.
\end{aligned}$$

Using this formula we find

$$c_{n-1} - c_n = \frac{2}{(n+1)(n+2)} ((n+2)\mathcal{H}_n - (n+1)\mathcal{H}_{n+1}) = \frac{2(\mathcal{H}_n - 1)}{(n+1)(n+2)}$$

which proves the lemma. \square

Proof of Corollary 1.3. It is well-known that if $F(t)$ is a continuous positive definite function on \mathbb{R} , then $F(0) - F(t)$ is continuous and negative definite, and a continuous negative definite function $H(t)$ satisfies $\Re H(t) \geq H(0) \geq 0$, see [6]. Therefore (ii) follows from (i), and (iii) follows from (ii) because $G(a) \geq 0$ for $a \geq 1$. \square

Proof of Proposition 1.4. By (i) and (iii) of Lemma 2.1 it is clear that

$$\int_{-\infty}^{\infty} |1 - G(it)|^2 \frac{dt}{2\pi} < \infty$$

and that dominated convergence can be applied to yield the first equality in (15). By Plancherel's theorem $1 - G(it)$ must be the Fourier transform of a square integrable function, which is the L^2 -limit of (13), hence equal to $\varphi(s)1_{[0,\infty)}(s)$. \square

Before proving Proposition 1.5 we give Lemma 2.6.

Lemma 2.6 *For $z \in S(0, \pi/3)$ we have $\Im G'(z) < 0$.*

Proof. From the relation (8) it follows that

$$\Im G'(re^{i\theta}) = -2r \sin \theta \int_0^{\infty} \frac{r \cos \theta + t}{((r \cos \theta + t)^2 + (r \sin \theta)^2)^2} \rho(t) dt. \quad (28)$$

We claim that for fixed $r > 0$ and $\theta \in [0, \pi/3]$ the function

$$k(t) = \frac{r \cos \theta + t}{((r \cos \theta + t)^2 + (r \sin \theta)^2)^2}$$

is decreasing. Indeed, it follows that

$$k'(t) = \frac{(r \sin \theta)^2 - 3(r \cos \theta + t)^2}{((r \cos \theta + t)^2 + (r \sin \theta)^2)^3}$$

and the numerator is negative for all $t > 0$ because $\sin^2 \theta \leq 3 \cos^2 \theta$ for $\theta \in [0, \pi/3]$.

This implies

$$\begin{aligned} \int_0^{\infty} \frac{r \cos \theta + t}{((r \cos \theta + t)^2 + (r \sin \theta)^2)^2} \rho(t) dt &= \int_0^{t_0} k(t) \rho(t) dt + \int_{t_0}^{\infty} k(t) \rho(t) dt \\ &\geq k(t_0) \left(\int_0^{t_0} \rho(t) dt + \int_{t_0}^{\infty} \rho(t) dt \right), \end{aligned}$$

where $t_0 = (\sqrt{5} - 1)/2$. From Lemma 2.3 it follows that the integral above is positive. From (28) we now obtain that

$$\Im G'(re^{i\theta}) = -2r \sin \theta \int_0^\infty \frac{r \cos \theta + t}{((r \cos \theta + t)^2 + (r \sin \theta)^2)^2} \rho(t) dt < 0.$$

This proves the lemma. □

Proof of Proposition 1.5. From (8) it follows that

$$\Im G(x + iy) = y \int_0^\infty \frac{\rho(t)}{(x + t)^2 + y^2} dt.$$

Here $t \mapsto 1/((x + t)^2 + y^2)$ is a decreasing function of t and it follows as in Lemma 2.6 that $\Im G(x + iy) > 0$ for $x > 0$ and $y > 0$ and also $\Im G(x + iy) < 0$ for $x > 0$ and $y < 0$. Hence it is enough to show that G is one-to-one in the sector $S(0, \pi/3)$.

For z_1 and z_2 belonging to the sector $S(0, \pi/3)$ we have

$$G(z_2) - G(z_1) = \int_{\gamma(z_1, z_2)} G'(w) dw,$$

where $\gamma(z_1, z_2)$ is the straight line segment from z_1 to z_2 . Thus

$$G(z_2) - G(z_1) = (z_2 - z_1) \int_0^1 G'(z_1 + t(z_2 - z_1)) dt \neq 0,$$

when $z_1 \neq z_2$ since $\Im G'(w) < 0$ for $w \in S(0, \pi/3)$ by Lemma 2.6. This shows that G is one-to-one in $S(0, \pi/3)$. □

References

- [1] H. Alzer, *Inequalities for the Volume of the Unit Ball in \mathbb{R}^n , II*, Mediterr. j. math. **5** (2008), 395–413.
- [2] G. D. Anderson, S.-L. Qiu, *A monotonicity property of the gamma function*, Proc. Amer. Math. Soc. **125** (1997), 3355–3362.
- [3] G. D. Anderson, M. K. Vamanamurthy and M. Vuorinen, *Special functions of quasiconformal theory*, Expo. Math. **7** (1989), 97–136.
- [4] C. Berg, *Quelques remarques sur le cône de Stieltjes*. In: *Séminaire de Théorie du potentiel, Paris n.5*. Lecture Notes in Mathematics **814**. Springer-Verlag, Berlin-Heidelberg-New York, 1980.

- [5] C. Berg, *Stieltjes-Pick-Bernstein-Schoenberg and their connection to complete monotonicity*. In: Jorge Mateu and Emilio Porcu, Positive Definite Functions: From Schoenberg to Space-Time Challenges. Department of Mathematics, Universitat Jaume I, Castelló de la Plana 2008.
- [6] C. Berg, G. Forst, Potential Theory on Locally Compact Abelian Groups, Springer, Berlin-Heidelberg-New York, 1975.
- [7] C. Berg, H. L. Pedersen, *A completely monotone function related to the Gamma function*, J. Comput. Appl. Math. **133** (2001), 219–230.
- [8] C. Berg, H. L. Pedersen, *Pick functions related to the gamma function*, Rocky Mount. J. Math. **32** (2002), 507–525.
- [9] C. Berg, H. L. Pedersen, *A one parameter family of Pick functions defined by the Gamma function and related to the volume of the unit ball in n -space*. Proc. Amer. Math. Soc. **139** no. 6 (2011), 2121–2132.
- [10] A. Elbert, A. Laforgia, *On some properties of the Gamma function*, Proc. Amer. Math. Soc. **128** (2000), 2667–2673.
- [11] Feng Qi, Bai-Ni Guo, *Monotonicity and logarithmic convexity relating to the volume of the unit ball*, arXiv:0902.2509v1[math.CA].
- [12] G. E. H. Reuter, *Über eine Volterrasche Integralgleichung mit totalmonotonem Kern*, Arch. Math. **7** (1956), 59–66.
- [13] D. V. Widder, *The Laplace Transform*. Princeton University Press, Princeton, 1941.